ASYMPTOTIC AXISYMMETRIC SWIRLING JET OF IDEAL INCOMPRESSIBLE FLUID

(ASIMPTOTIKA OSESIMMETRICHNOI ZAVIKHRENNOI STRUI IDEAL'NOI NESZHIMAEMOI ZHIDKOSTI)

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The behavior of axisymmetric swirling jets at large distances from their place of origin is considered. In certain cases the occurrence of surface waves (periodic expansion and contraction of the jet) is possible. The analogous fact for plane jets was noticed in [1, 2].

1. If we introduce for consideration the stream function ψ then, taking for the independent variables the quantities (x, ψ, θ) and for the required function the quantity y, where x is the distance along the axis of symmetry, y the distance from the axis of symmetry, and θ the polar angle, we obtain for axisymmetric flows of an ideal incompressible liquid the system [3]:

$$u_x = \frac{1}{yy_\psi}, \qquad u_y = -\frac{y_x}{yy_\psi}, \qquad u_\theta = \frac{1}{y}\,\Phi(\psi) \tag{1.1}$$

$$\frac{y}{y_\psi}\frac{\partial}{\partial\psi}\left(\frac{1}{yy_\psi}\right) + \frac{y_x}{y_\psi}\frac{\partial}{\partial\psi}\left(\frac{y_x}{y_\psi}\right) - \frac{\partial}{\partial x}\left(\frac{y_x}{y_\psi}\right) + \Phi\Phi' - y^2F' = 0 \tag{1.2}$$

$$\frac{1}{2y^2}\left(\frac{1+y_x^2}{y_\psi^2} + \Phi^2\right) - F + \frac{p}{\rho} = 0 \tag{1.3}$$

Here u_x, u_y and u_θ are the components of the velocity vector, and F and Φ are arbitrary functions of the one variable ψ, characterizing the specific energy and circulation of the flow.

The boundary conditions are as follows.

On the axis of the jet

$$y = 0, \qquad \psi = 0 \tag{1.4}$$

If we consider the motion in dimensionless variables, taking for the unit of length a certain characteristic thickness of the jet h and the quantity $\psi_0 = Q / 2\pi$, where Q is the discharge of the fluid, then on the free surface we shall have $\psi = 1$. Moreover, the pressure at the surface of the jet is constant, and without loss of generality can be taken as $p = 0$. Then (1.3) gives

$$\frac{1+y_x^2}{y_\psi^2}+\Phi^2-2y^2F=0 \quad \text{when } \psi=1 \tag{1.5}$$

2. At large distances from the place of origin we can assume that the jet has the form of a circular cylinder. For such a flow the required quantities do not depend upon x

$$y=t\,(\psi), \qquad u_x=u\,(\psi) \tag{2.1}$$

To determine the functions u and t from (1.2) and (1.3) we have the system

$$u=\frac{1}{tt_\psi}, \quad uu'-F'+\frac{\Phi\Phi'}{t^2}=0, \quad t\,(1)=1, \quad t\,(0)=0 \tag{2.2}$$

Integrating the second equation (2.2), we obtain

$$\frac{u^2}{2}-F+\frac{1}{2}\frac{\Phi^2}{t^2}+\int_1^t\frac{\Phi^2}{t^3}\,dt=0 \tag{2.3}$$

From (1.3) and (2.8) the pressure is determined by the formula

$$p=-\int_t^1\frac{\Phi^2}{t^3}\,dt \tag{2.4}$$

From this we draw the following conclusions. In an irrotational jet ($\Phi\equiv 0$) the pressure is constant. In a swirling jet the pressure at the axis of the jet is always less than the pressure at its periphery. This is confirmed by the formation of eddies at the centre of the jet.

At the centre of the jet the velocity u_θ and the pressure can tend to infinity; in particular, when $\Phi\,(0)\neq 0$. Such flows will be known as flows with axial vortices. For $\Phi=\text{const}$, $u=0$ and $F'=0$ we shall have a rectilinear threadlike vortex.

Let us write down again the solution corresponding to an irrotational jet $F'=\Phi=0$. In the dimensionless variables adopted it has the form

$$u=2, \quad t=(\psi)^{1/2}, \quad F=2, \quad \Phi=0, \quad p=0 \tag{2.5}$$

3. In what follows it is convenient to take as the independent variable $t=t\,(\psi)$ and set

$$y=t+z\,(t,\,x) \tag{3.1}$$

The equation for $z\,(t,\,x)$ and the boundary conditions (1.3)–(1.5) can be written thus[1]:

$$\frac{z-tz_t}{1+z_t}\frac{\partial u}{\partial t}-\frac{t+z}{1+z_t}\frac{\partial}{\partial t}\frac{u\,(z+tz_t+zz_t)}{(t+z)\,(1+z_t)}+\frac{z_x}{1+z_t}\frac{\partial}{\partial t}\frac{utz_x}{1+z_t}-$$
$$-\frac{\partial}{\partial x}\frac{utz_x}{1+z_t}-\frac{2zt+z^2}{ut}F_t'=0, \quad z\,(0)=0 \tag{3.2}$$
$$\frac{u^2\,(-2z_t-z_t^2+z_x^2)}{(1+z_t)^2}-2\,(2z+z^2)\,F_0-2F_1=0 \quad \text{when } t=1$$

[1] Here we have made the substitution $F\,(1)=F_0\,(1)+F_1$; where $F_0\,(1)$ corresponds to a jet having the form of a circular cylinder. Two jets with identical distributions of velocity (u) and vorticity (F' and Φ) at a certain section have different specific energy F according to the shape of the free surface of the jet. From what follows it will be seen that $F_1=O\,(z^2)$.

4. Linearisation of the equations and boundary conditions (3.2) gives

$$\frac{\partial}{\partial t}(u^2tz_t) - \left(\frac{u^2}{t} - 2F'\right)z = -u^2tz_{xx},\ z(x,0) = 0,\ z_t + \frac{2F}{u^2}z = 0 \text{ when } t = 1 \quad (4.1)$$

We shall seek the eigenfunctions of (4.1) by separating the variables. Setting $z = T(t)X(x)$, we obtain

$$LT \equiv \frac{d}{dt}(u^2tT') + \left[\mu u^2t - \left(\frac{u^2}{t} - 2F'\right)\right]T = 0 \quad (4.2)$$

$$T(0) = 0, \quad T'(1) + \frac{2F(1)}{u^2(1)}T(1) = 0 \quad (4.3)$$

$$X'' - \mu X = 0$$

Having a complete set of eigenfunctions of the problem (4.1) we can obtain in the linear formulation the solution of the problem of a jet with a given velocity distribution at two sections of the jet (one for the semibounded jet). The behavior of the jet depends on the nature of the spectrum of the boundary problem (4.2).

If all the eigenvalues μ of the problem (4.2) are positive, then, as is obvious from (4.3), the semibounded jet either expands without limit downstream (if it has expanded to a certain point, sufficiently far from the place of origin), or else it will contract, tending asymptotically to take a cylindrical form. The influence of conditions at the exit decays exponentially downstream. The asymptotic free surface of the jet as $x \to \infty$ in the case of a discrete spectrum has the form

$$y = 1 + C\exp(-\sqrt{\mu_0}x) + O(\exp(-\sqrt{\mu_1}x)) \qquad (\mu_1 > \mu_0 > 0)$$

Accordingly, the index of exponential contraction of the jet is equal to $\sqrt{\mu_0}$, where μ_0 is the smallest positive eigenvalue[1] of the problem (4.2).

It is clear from (4.3) that wave modes do not arise in flows for which all the eigenvalues of (4.2) are positive. The presence in the spectrum of negative eigenvalues, however, leads to the occurrence of wave modes. Each negative eigenvalue will play the role of a critical number[2]. Solutions corresponding to positive eigenvalues decrease rather rapidly, and at large distances from the origin the form of the jet is determined by the negative spectrum of the problem (4.2). If the negative spectrum is finite, then it cannot substantially change the form of the jet, inducing only sinusoidal changes in the form of the surface. If, however, the negative spectrum is unbounded, then the shape of the jet can be greatly altered.

5. The boundary problem (4.2) is the Sturm-Liouville problem for the second order self-conjugate differential equation. It is easy to show that all of its eigenvalues are real. When $F' \leqslant 0$ they are all strictly positive, and wave modes do not occur in the jet.

Negative eigenvalues (wave modes) can take place only when $F' > 0$ (the specific energy increases from the centre to the periphery of the jet). If μ is not an eigenvalue of the homogeneous boundary problem (4.2), then the inhomogeneous boundary problem

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[1] As is shown below, in the case when F' and u^2 are meromorphic functions of t (with a singular point when $t = 0$), the spectrum of the problem (4.2) is discrete.

[2] In plane irrotational flow the same role is played by the number $c/\sqrt{gh}$ (c is the velocity of wave propagation, g is the acceleration due to gravity and h is the depth of the stream).

$$LT = \Phi_1 (t, x), \quad T (0) = 0, \quad T' (1) + \frac{2F (1)}{u^2 (1)} T (1) = \Phi_2 (x) \qquad (5.1)$$

is always soluble; if however $\mu = \mu_n$ is an eigenvalue (simple), then the condition for solubility of the boundary problem (5.1) is

$$\int_0^1 \Phi_1 (x, t)\, T_n (t)\, dt - u^2 (1)\, T_n (1)\, \Phi_2 (1) = 0 \qquad (5.2)$$

where $T_n(t)$ is the eigenfunction of the problem (4.2) corresponding to the eigenvalue $\mu = \mu_n$.

For a jet with an axial eddy the specific energy F at the axis of the jet can tend to infinity.[1] We shall therefore assume that u^2 and F' are meromorphic functions which in the neighborhood of $t = 0$ can be expanded in a Laurent series

$$u^2 = \sum_{k=n}^{\infty} a_k t^k, \qquad F' = \sum_{k=m}^{\infty} f_k t^k \qquad (a_n > 0)$$

(m and n can be negative).

Let us introduce the following results without proof. The spectrum of problem (4.2) is discrete. If $m < n - 1$, then for $f_m < 0$ (the specific energy decreases from the axis of the jet) there is a finite number of negative eigenvalues, and for $f_m > 0$ the spectrum contains an infinite number of negative eigenvalues. If $m \geqslant n - 1$, then writing $c = {}^1/_4 (n^2 + 3) - 2f_{n-1} / a_n$, we have for $c \geqslant -{}^1/_4$ a finite number of negative eigenvalues, whilst for $c < -{}^1/_4$ the spectrum is unbounded on both sides. (The latter is possible only for flows in which the longitudinal velocity u_x at the axis of the jet tends to infinity). The requirement $T (0) = 0$ gives the further condition that $2f_{n-1} / a_n \leqslant 1$.

6. The equations (3.2) enable us to determine waves of small amplitude on the surface of the swirling jet. The fact that vortices can be the reason for the appearance of waves was noted by M.A. Lavrent'ev and N.N. Moiseev [2, 4]. Waves arise on the surface of the jet in the case when (4.2) has negative eigenvalues. At very great distances from the place of origin the exponentially decaying terms can be neglected, and the jet will have an expression of a wave-like profile. Let us consider an actual example

$$u = 2, \qquad \psi = t^2, \qquad \Phi = 2\alpha, \qquad F = -k (1 - t^2) + 2\alpha^2 + 2 \qquad (6.1)$$

The equations and boundary conditions of section 2 are satisfied. With $\alpha = k = 0$ we obtain an irrotational jet (2.5). The equation (4.1) for the given case has the form

$$\frac{\partial}{\partial t} (t z_t) - \left(\frac{1}{t} - kt\right) z + t z_{xx} = 0 \qquad (6.2)$$

The substitution $z = e^{-\mu x} T (t)$ gives for the determination of T (6.3)

$$\frac{1}{t} (tT')' + \left[(k + \mu^2) - \frac{1}{t^2}\right] T = 0, \qquad T (0) = 0, \qquad T' (1) + (\alpha^2 + 1)\, T (1) = 0$$

The general solution of (6.2) has the form

$$z = \sum_{n=1}^{\infty} (A_n e^{\mu_n x} + B_n e^{-\mu_n x})\, J_1 (\sqrt{\mu_n^2 + k}\, t), \qquad t = \psi^{1/2}, \qquad \mu_n^2 = s_n^2 - k \quad (6.4)$$

Here J_1 is the Bessel function, and s_n is a positive root of the equation

$$s J_1' (s) + (\alpha^2 + 1)\, J_1 (s) = 0 \qquad (6.5)$$

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[1] For example, for a circular vortex with the distribution

$$u_x = 0, \quad \Phi = p_0 + p_1 t, \quad u_\theta = p_0 / t + p_1, \quad F' = -p_0 p_1 / t^2 - p_1^2 / t, \quad p_0 = \Gamma / 2\pi$$

The coefficients A_n and B_n can be determined, knowing the distribution of velocities ($z = g_i(t)$) at two sections of the jet when $x = x_0$ and $x = x_1$.

Waves arise on the surface of the jet when $\mu_m < 0$ ($k > s_m^2$) and have the wavelength

$$\lambda_m = 2\pi / \sqrt{k - s_m^2}$$

If, however, $k < s_0^2$, where s_0 is the smallest root of (6.5), then for unbounded extension of the jet it is necessary to set $A_n = 0$, then B_n are determined from the distribution $z = g_0(t)$ at a certain section of the jet. For sufficiently large $x = x_0$ we can obtain an asymptotic representation of the form of the jet, substituting in (6.4) only the term corresponding to the smallest of the roots s_0. The free surface is accordingly described to a high degree of accuracy by the equation[1]

$$y = 1 + B \exp(-\sqrt{s_0^2 - k}x)$$

For the irrotational jet $\alpha = k = 0$ and, moreover, (6.5) simplifies to $J_0(s)=0$, which agrees with the result obtained in [6], where several examples were considered in detail of the actual representation of Φ and F in the case when the spectrum of the problem (4.2) is positive (waves absent from the surface of the jet).

7. One of the interesting cases of wave modes on the surface of a jet is that of long waves. For long waves we can obtain an asymptotic solution of problem (3.2), using the method of [1]. It is applicable to the given case if (4.2) has $\mu = 0$ as an eigenvalue. Changing the circulation of the flow we can always achieve this. In fact, with $\mu = 0$ and specified u^2 and F', taking for T_0 a particular solution of (4.2) satisfying the first boundary condition, we can select $F(1)$ so as to fulfil the second condition also. At the same time, we change the circulation Φ by the quantity Φ_1 so as not to change u and F'. For this it is sufficient to ensure that $1 + 2\Phi / \Phi_1 = C \exp(-\Phi_1^2)$. This equation has a root for any C. We can say therefore that the circulation of the flow is the cause of the change in the form of the jet. In the case considered in section 6, $\mu = 0$ is an eigenvalue of problem (4.2) if we set $k = s^2$.

Let us introduce in problem (3.2) a small parameter, setting $\xi = \sqrt{\varepsilon}x$, and expand $z(\xi, t)$ in the form of a power series in ε

$$z = \sum_{k=1}^{n} \varepsilon^k z_k(\xi, t), \qquad F(1) = F_0(1) + \varepsilon^2 F_1$$

The quantity ε characterizes the deviation of the energy F from the energy of a cylindrical jet F_0. To a first approximation we obtain

$$z_1 = C(\xi) T_0(t) \tag{7.1}$$

Here $T_0(t)$ is the eigenfunction of problem (4.2) corresponding to the eigenvalue $\mu = 0$.

For the second approximation we have

$$L(z_2) = \Phi_1(z_1), \quad z_2(0) = 0, \quad z_2'(1) + \frac{2F_0(1)}{u^2(1)} z_2(1) = \Phi_2(z_1) \tag{7.2}$$

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[1] A similar method for the plane irrotational jet gives for the exponent of the exponential contraction of the jet the value $s_0 = \pi$, as follows from the Mitchell-Zhukovski formulae [3].

Here

$$\Phi_1(z) = -u^2tz_{\xi\xi} + g(z), \quad \Phi_2(z) = {}^3/_2 z_t^2 - \frac{z^2}{u^2}F_0 - \frac{1}{u^2}F_1$$

$$g(z) = -\frac{z^2}{t}F_t' + 3tz_t^2uu' + \frac{u^2}{t^2}(-z^2 - tzz_t + 2tz_t^2 + 3t^2z_tz_{tt}) \qquad (7.3)$$

Substituting in (7.3) the quantity z_1 from (7.1), we obtain from the condition of solubility (5.2) of the inhomogeneous boundary problem an equation determining $C(\xi)$.

Let us introduce the symbols γ and p by the following relations

$$\gamma = -{}^1/_2\int_0^1 u^2tT_0^2dt, \qquad \pm 3\gamma p^2 = \int_0^1 g(T_0)\,T_0dt - \left(\frac{6F_0^2}{u^4} - F_0\right)T_0^3(1)$$

($\gamma < 0$, and $T_0(1)$, without loss of generality, can be taken as positive, and the sign on the left-hand side can be chosen as the sign on the right). Moreover, let us choose F_1 to satisfy the relations $T_0(1)F_1 = \mp 4\gamma p^2$.

For the determination of $C(\xi)$ we shall have the equation

$$2C'' \pm 3p^2C^2 \mp 4p^2 = 0 \qquad (7.4)$$

having the solution

$$C = \pm 2\,\mathrm{Cn}^2(p\xi, 1/\sqrt{2})$$

Transforming back to the old variables, with an accuracy of $O(\varepsilon^2)$ we obtain

$$y = 1 \pm 2\varepsilon T_0(t)\,\mathrm{Cn}^2(p\sqrt{\varepsilon}x, 1/\sqrt{2}) \qquad (7.5)$$

The free surface is obtained by setting $t = 1$ in (7.5).

Accordingly, a one-parameter family of flows is obtained; the parameter of the flow is related to the amplitude of the wave a by the relation $\varepsilon = a / 2T_0(1)$. (We notice that for gravitational waves in the plane case there is a two-parameter family of flows [1, 2]). The length of the wave, if by this we understand the distance between two adjacent crests (troughs), is determined by the formula

$$\lambda = \frac{1}{p\sqrt{\varepsilon}}K\left(\frac{1}{\sqrt{2}}\right) \approx \frac{1.85}{p\sqrt{\varepsilon}}, \qquad \text{or} \quad \lambda \approx \frac{1.85}{p}\left(\frac{2T_0(1)}{a}\right)^{1/2} \qquad (7.6)$$

Comparing (7.8) with the condition determining F_1 we see that if for the characteristic dimension we choose the minimum thickness of the jet (upper sign), then $F_1 > 0$, i.e., the specific energy of the wave motion is greater than the specific energy of the parallel flow with the same F', u^2 and thickness of jet. If for the characteristic dimension we take the maximum thickness of the jet, then $F_1 < 0$ and the specific energy of the wavy jet is less. The difference in the specific energies is of order a^2.

When $p = 0$ equation (7.4) has no periodic solution. This does not signify, however, that no flow modes exist other than trivial ones. In this case it is necessary to seek the solution of (3.2) using fractional powers of ε, applying the method of [7].

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